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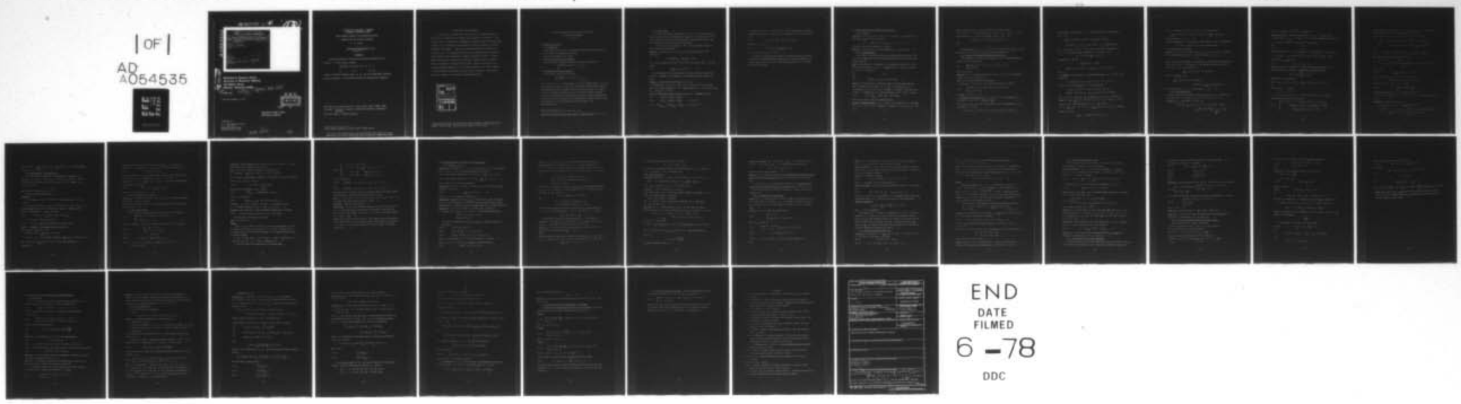

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P. L. Lions

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SOME PROBLEMS RELATED TO THE BELLMAN-DIRICHLET EQUATION FOR TWO ELLIPTIC OPERATORS

P. L. Lions*

Technical Summary Report # 1816
January 1978

ABSTRACT

We prove existence, uniqueness and regularity properties for a solution u of the control problem:

$$\begin{cases} \max(A_1u-f_1,\ A_2u-f_2,\ u-\psi) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\ ; \end{cases}$$

as well as related problems, where A_1, A_2 are two second-order uniformly elliptic operators. Our methods are based upon monotonicity arguments.

AMS (MOS) Subject Classifications: 35J25, 35J60, 35K20, 35K60, 49A20

Key Words: Stochastic integrals, Variational inequality, Regularity estimates

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

In this report we study some special cases of the Bellman equation which arises in many applied contexts such as physics and economics. This equation may be considered as a boundary value problem where the differential operator is the supremum (or the infimum) of several second-order elliptic operators (for example). The special cases studied here are mainly the following ones: 1) A problem with two elliptic operators and a constraint; this problem can be interpreted in terms of stochastic control: at each moment we choose one (between two) diffusion process, and at any time we have the possibility of stopping the process. Then the solution of the problem considered minimizes a cost function related to the control chosen. 2) A problem with one parabolic and one elliptic operator: this problem arises in economics as a problem of optimal maintenance.

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SOME PROBLEMS RELATED TO THE BELLMAN-DIRICHLET EQUATION

FOR TWO ELLIPTIC OPERATORS

P. L. Lions*

I. Introduction and results:

I.1. Introduction:

In this paper we prove existence, uniqueness, and regularity theorems for a solution of special cases of the Bellman-Dirichlet problem.

Let us give two examples of our results:

1) Two elliptic operators and one constraint:

We prove existence, uniqueness and regularity theorems for a solution of:

$$(I.1)\qquad \begin{cases} \max(A_1u-f_1,\ A_2u-f_2,\ u-\psi) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma\ . \end{cases}$$

2) One parabolic and one elliptic operator:

We prove similar results for a solution of:

$$(I.2)\qquad \begin{cases} \max(\frac{\partial u}{\partial t} + A_1u - f_1,\ A_2u - f_2) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma\ . \end{cases}$$

In (I.1) and in (I.2) Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary Γ, f_1, f_2, ψ are given functions, and A_1, A_2 are linear, second order, uniformly elliptic operators. These problems are related to problems of control of diffusion processes. For example: (I.1) may be considered as a problem of control of stochastic integrals with optimal stopping. These interpretations and analytical results using stochastic representations of solutions of problems considered here will be developed in a subsequent paper to appear. Let us mention that (I.2) is a problem which arises in economics: see Bensoussan-Lesourne [1].

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I.2. Known results:

1) In the case of the whole space, Krylov in [4], [5], [6], [7] has obtained very general results on the general Bellman-Dirichlet equation. The proofs are very delicate and use essentially very precise results on stochastic integral theory; moreover, they do not seem to give results in the case of bounded domains.

2) In the case of a bounded domain (in $\mathbb{R}^n$) with smooth boundary, Brezis and Evans [2] have the following result.

Theorem I.1. [Brezis-Evans]: Let A_1, A_2 be two elliptic operators with $C^2(\bar{\Omega})$ coefficients:

$$A_i u = -a^i_{kj}(x)u_{x_k x_j} + b^i_k(x)u_{x_k} + c^i(x)u \ .$$

If μ is large enough, then for each $f^1, f^2 \in L^2(\Omega)$, there exists a unique $u \in H^2 \cap H^1_0$ solving:

$$\max_{i\in\{1,2\}} \{A_i u + \mu u - f^i\} = 0 \ . \qquad \blacksquare \tag{I.3}$$

In the following, we shall always consider such elliptic operators, unless stated otherwise. Furthermore, for the problem I.3; the following results of regularity are known:

Theorem I.2: Under assumptions of Theorem I.1 on A_1, A_2, μ

1) [Brezis-Evans]: If $f_1, f_2 \in H^1$, then $u \in H^3$. If $f_1, f_2 \in W^{1,p}(p>n)$, then for each $\Omega' \subset\subset \Omega$ there exists some $0 < \alpha < 1$, depending only on p, Ω' and the coefficients of A_i such that $u \in C^{2,\alpha}(\bar{\Omega}')$.

[And in all Brezis-Evans results stated here, we have: $\exists c$ independent of f_1, f_2

$$\|u\|_{H^2} \leq c\{\|f_1\|_{L^2} + \|f_2\|_{L^2}\} \ , \tag{I.4}$$

$$\|u\|_{H^3} \leq c\{\|f_1\|_{H^1} + \|f_2\|_{H^1}\} \quad \text{if } f_1, f_2 \in H^1 \ , \tag{I.5}$$

$$\|u\|_{C^{2,\alpha}(\bar{\Omega}')} \leq c\{\|f_1\|_{W^{1,p}} + \|f_2\|_{W^{1,p}}\} \ , \text{ if } f_1, f_2 \in W^{1,p}] \ . \tag{I.6}$$

2) [Evans-Lions] [3]: If $f_1, f_2 \in L^n$, then $u \in C^0(\bar{\Omega})$ and there exists a constant c such that for two solutions u (resp. v) of I.3 corresponding to f_1, f_2 (resp. g_1, g_2) in L^n:

$$\|u - v\|_{C^0(\bar{\Omega})} \leq c\{\|f_1 - g_1\|_{L^n} + \|f_2 - g_2\|_{L^n}\} . \tag{I.7}$$

And we have if $f_i, g_i \in L^\infty$:

$$\|u - v\|_{C^0(\bar{\Omega})} \leq \frac{1}{\mu} \max_{i \in \{1,2\}} \|f_i - g_i\|_\infty . \tag{I.8}$$

If $f_1, f_2 \in W^{1,p}(p>n)$, then $u \in W^{1,\infty}(\Omega)$. ■

In Section II we study the Bellman equation for two operators and one constraint; in Section III we consider some other problems related to the Bellman equation for two elliptic operators; and in Section IV we study the Bellman equation for one parabolic and one elliptic operator.

II. Bellman equation for two operators and one constraint:

II.1. Main result:

Theorem II.1: Let $f_1, f_2 \in L^2$, $\psi \in H^2$, $\psi \geq 0$ on $\partial\Omega$ and let μ be large enough. Then there exists a unique $u \in H^2 \cap H_0^1$ solving:

$$\text{(I.1)} \qquad \max(A_1 u + \mu u - f_1,\ A_2 u + \mu u - f_2,\ u - \psi) = 0 . \qquad \blacksquare$$

The proof will be given in subsections II.3 and II.4.

Remark II.1: We shall see in III.1 how the assumption on μ may be given up. ■

II.2. Preliminary results:

In this section we state results which will be constantly used in this paper. The first one is a lemma, which was basic for Brezis-Evans proof [see a proof of this lemma in Brezis-Evans [2]].

Lemma II.1 (Sobolevsky [13], Ladyženskaja [8, §3], Ladyženskaja-Ural'ceva [9, p. 182]): There exists two constants $c_1 \geq 0$, $c_2 > 0$ depending only on Ω and the coefficients of A^1 and A^2, such that if

$$\text{(II.1)} \qquad \mu \geq c_1$$

then

$$\text{(II.2)} \qquad \|v\|^2_{H^2} \leq c_2 \int_\Omega (A_1 v + \mu v)\cdot(A_2 v + \mu v)dx . \qquad \blacksquare$$

Remark II.2: We shall need in the following a result not stated by Brezis-Evans but which is transparent in their proof. Let u (resp. v) be the solution of (I.3) corresponding to f_1, f_2 (resp. g_1, g_2) (we suppose μ big enough). Then there exists a constant c (depending only on Ω and the coefficients of A^1 and A^2) such that:

$$\text{(II.3)} \qquad \|u - v\|_{H^2(\Omega)} \leq c\{\|f_1 - g_1\|_{L^2} + \|f_2 - g_2\|_{L^2}\} . \qquad \blacksquare$$

A basic tool to solve I.1 will be the following result:

Theorem II.2 (Maximum principle): Let u (resp. v) be the solution of (I.3) corresponding to f_1, f_2 (resp. g_1, g_2) (we suppose μ big enough). If $f_1 \leq g_1$, $f_2 \leq g_2$, then

$$\text{(II.4)} \qquad u \leq v .$$

Proof: Suppose first that $f_1, f_2, g_1, g_2 \in W^{1,p}(p>n)$, then from Theorem I.2: $u, v \in C^{2,\alpha}(\Omega) \cap C^0(\bar{\Omega})$. Suppose now there exists a point x_0 in Ω where $v - u$ has a positive maximum: at x_0 we have (for example) $A_1 u(x_0) = f_1(x_0)$. Thus

$$A_1(v-u)(x_0) \leq 0, \quad \text{but} \quad A_1(v-u)(x_0) \geq \mu(v-u)(x_0) .$$

Thus

$$u \leq v .$$

Then approximating any f_1, f_2, g_1, g_2 in L^2 by $f_1^n, f_2^n, g_1^n, g_2^n$ which are in $W^{1,p}$ and such that: $f_1^n \leq g_1^n$, $f_2^n \leq g_2^n$, we get the result from Remark II.2. ■

II.3. Resolution of the penalized problem associated to (I.1):

To prove Theorem II.1, we shall build a solution u of (I.1) by approximating the problem (I.1) by the following one:

$$\text{(II.5)} \quad \begin{cases} \max(A_1 u_\varepsilon + \mu u_\varepsilon - f, A_2 u_\varepsilon + \mu u_\varepsilon - f_2) + \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ = 0 \\ u_\varepsilon \in H^2 \cap H_0^1 . \end{cases}$$

Thus we need to prove:

Theorem II.3: If μ is sufficiently large, if $f_1, f_2 \in L^2$ and $\psi \in H^1$. There exists a unique $u_\varepsilon \in H^2 \cap H_0^1$ solving (II.5).

Proof: The proof will be divided in several steps.

1) Prop. II.1: If $f_1, f_2 \in L^\infty$, $\psi \in L^\infty$, there exists a unique u_ε solving

$$\text{(II.5')} \quad \begin{cases} \max(A_1 u_\varepsilon + \mu u_\varepsilon - f_1, A_2 u_\varepsilon + \mu u_\varepsilon - f_2) + \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ = 0 \\ u_\varepsilon \in L^n \cap H^2 \cap H_0^1 . \end{cases}$$

Furthermore $u_\varepsilon \in C^0(\bar{\Omega})$. ■

2) Existence of a solution of (II.5): Let $f_1, f_2 \in L^2$, $\psi \in H^1$, approximate f_1, f_2 by $f_1^n, f_2^n \in L^\infty$, ψ by $\psi^n \in L^\infty$: $f_i^n \xrightarrow[L^2]{} f_i$, $\psi^n \xrightarrow[H^1]{} \psi$.

*We are going to prove first that u_ε^n the solution of (II.5') corresponding to f_i^n, ψ^n is bounded in H^2: we have

$$\int_\Omega \{A_1 u_\varepsilon^n - f_1^n + \mu u_\varepsilon^n + \frac{1}{\varepsilon}(u_\varepsilon^n - \psi)^+\}, \{A_2 u_\varepsilon^n - f_2^n + \mu u_\varepsilon^n + \frac{1}{\varepsilon}(u_\varepsilon^n - \psi)^+\} dx = 0 .$$

Then, by Lemma II.1 and denoting $a_i(\cdot,\cdot)$ the bilinear forms on H_0^1 associated to $A_i + \mu$: $\exists c_2$

$$c_2\|u_\varepsilon^n\|_{H^2}^2 + \frac{1}{\varepsilon} a_1((u_\varepsilon^n-\psi^n)^+, (u_\varepsilon^n-\psi^n)^+) + \frac{1}{\varepsilon} a_2((u_\varepsilon^n-\psi^n)^+, (u_\varepsilon^n-\psi^n)^+)$$

$$+ \frac{1}{\varepsilon^2}\|(u_\varepsilon^n-\psi)^+\|_{L^2}^2 \leq -\frac{1}{\varepsilon}\sum_{i=1,2} a_i(\psi^n, (u_\varepsilon^n-\psi^n)^+) + \sum_{i=1,2}\int_\Omega f_i^n\{A_i u_\varepsilon^n + \mu u_\varepsilon^n\}dx \ .$$

If μ is big $a_1(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$ are coercive: thus (u_ε^n) is bounded in H^2.

*Let u^n be a subsequence weakly convergent in H^2 to some u_ε: then

$$f_i^n - \frac{1}{\varepsilon}(u_\varepsilon^n - \psi^n)^+ \xrightarrow[L^2]{} f_i - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ \ .$$

From Remark II.2: we have

$$u_\varepsilon^n \xrightarrow[H^2]{} v \quad \text{where } v \text{ solves :}$$

$$\max(A_1 v + \mu v - f_1 + \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+, A_2 v + \mu v - f_2 + \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+) = 0 \ .$$

But $v = u_\varepsilon$.

3) <u>Uniqueness</u>: Let us denote by T the following application: if $u \in L^2$, Tu is the solution of

$$\frac{1}{\varepsilon} Tu + \max(A_1 Tu + \mu Tu - f_1, A_2 Tu + \mu Tu - f_2) = \frac{1}{\varepsilon}\psi - \frac{1}{\varepsilon}(u - \psi)^- \ .$$

Now we set $u^0 = \psi$ and we consider $u^n = T^n u^0$ by definition

$$\max(A_1 u_1 + \mu u_1 + \frac{1}{\varepsilon} u_1 - f_1, A_2 u_1 + \mu u_1 + \frac{1}{\varepsilon} u_1 - f_2) = \frac{1}{\varepsilon}\psi$$

$$\max(A_1 u_2 + \mu u_2 + \frac{1}{\varepsilon} u_2 - f_1, \quad A_2 u_2 + \mu u_2 + \frac{1}{\varepsilon} u_2 - f_2) = \frac{1}{\varepsilon}\min(\psi, u_1) \ .$$

But $f_i + \frac{1}{\varepsilon}\psi \geq f_i + \frac{1}{\varepsilon}\min(\psi,u)$, thus by Theorem II.2, $u_2 \leq u_1$. It is now easy to deduce $u^n \downarrow$ and $\max\limits_{i\in\{1,2\}} (A_i u^n + \mu u^n + \frac{1}{\varepsilon} u^n - f_i) \downarrow$ [for $n \geq 1$]. But let v_ε be a solution of II.5, by the same reasoning $u_1 \geq v_\varepsilon$. And we deduce $u^n \geq v_\varepsilon$, $\max\limits_{i\in\{1,2\}} (A_i u^n + \mu u^n + \frac{1}{\varepsilon} u^n - f_i) \geq \max\limits_{i\in\{1,2\}} (A_i v_\varepsilon + \mu u_\varepsilon + \frac{1}{\varepsilon} v_\varepsilon - f_i)$. Now with Remark II.2, we have

$$u^n \xrightarrow[H^2]{} u_\varepsilon \quad \text{a solution of II.5, and } u_\varepsilon \geq v_\varepsilon \ .$$

Thus u_ε is a maximum solution of II.5. But if v_ε is another solution

$$\max_{i\in\{1,2\}} \{A_i u_\varepsilon + \mu u_\varepsilon - f_i\} = -\frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ \leq -\frac{1}{\varepsilon}(v_\varepsilon - \psi)^+ = \max_{i\in\{1,2\}} \{A_i v_\varepsilon + \mu u_\varepsilon - f_i\} ,$$

then with Theorem II.2, $u_\varepsilon \leq v_\varepsilon$. ∎

Proof of Proposition II.1: It is obvious, in view of Theorem I.2, that T satisfies to:

$$\|Tu - Tv\|_\infty \leq \frac{1}{1+\varepsilon\mu} \|u - v\|_\infty ; \text{ and the result follows.} \quad ∎$$

The following result will be useful:

Prop. II.2: Let $f_1, f_2 \in L^2$ (resp. g_1, g_2), $\psi \in H^1$ (resp. φ) and let u_ε (resp. v_ε) be the corresponding solution of II.5. Then if we suppose $g_i \leq f_i$, $\varphi \leq \psi$

$$v_\varepsilon \leq u_\varepsilon .$$

Proof: From the proof of Theorem II.3 (existence part), we only need to prove the result for $f_1, f_2, g_1, g_2, \varphi, \psi \in L^\infty$. We denote by T_2 the application, which is contractive, defined by: $v = T_2 u$ is the solution of

$$\max_{i\in\{1,2\}} (A_i v + \mu v + \frac{1}{\varepsilon} v - f_i) = \frac{1}{\varepsilon}\min(u,\varphi) .$$

Then $u^n = T_2^n u_\varepsilon \xrightarrow[L^\infty]{} v_\varepsilon$, but

$$\max_{i\in\{1,2\}} (A_i u^1 + \mu u^1 + \frac{1}{\varepsilon} u^1 - g_i) = \frac{1}{\varepsilon}\min(u_\varepsilon,\varphi)$$

$$\leq \frac{1}{\varepsilon}\min(u_\varepsilon,\psi) = \max_{i\in\{1,2\}} (A_i u_\varepsilon + \mu u_\varepsilon + \frac{1}{\varepsilon} u_\varepsilon - f_i) .$$

Then from Theorem II.2: $u_\varepsilon \geq u^1$; and by induction $u_\varepsilon \geq u^n$. ∎

II.4. Proof of Theorem II.1:

First step: Existence of a solution: We are going to prove that u_ε converges in H^2 to a solution of I.1. We suppose $f_1, f_2 \in L^2$, $\psi \in H^2$ and $\psi \geq 0$ on $\partial\Omega$:

*Estimates on u_ε: Using Lemma II.1, we get: $\exists c_2$

$$c_2\|u_\varepsilon\|^2_{H^2} + \frac{1}{\varepsilon}\sum_{i=1,2} a_i((u_\varepsilon - \psi)^+, (u_\varepsilon - \psi)^+) + \frac{1}{\varepsilon^2}\|(u_\varepsilon - \psi)^+\|^2_{L^2}$$

$$\leq \sum_{i=1,2}\left(-A_i\psi, \frac{(u_\varepsilon - \psi)^+}{\varepsilon}\right)_{L^2} + \sum_{\substack{i=1,2\\ j=1,2\\ j\neq i}} (f_i, A_j u_\varepsilon) .$$

Then u_ε is bounded in H^2, $\frac{(u_\varepsilon - \psi)^+}{\varepsilon}$ is bounded in L^2.

*Passing to the limit: First remark that with Prop. II.2: $u_\varepsilon \downarrow$. Thus there exists $u \in H^2$ such that: $u_\varepsilon \downarrow u$, $u_\varepsilon \rightharpoonup u$ weakly in H^2. We extract a subsequence (still denoted by u_ε) such that $\frac{(u_\varepsilon - \psi)^+}{\varepsilon} \rightharpoonup \lambda$ weakly in L^2. Then we have:

$$\lambda \geq 0 \ , \tag{II.6}$$

$$u \leq \psi \ . \tag{II.7}$$

Furthermore on the set $A = \{u < \psi\}$ for almost $x \ \exists \ \varepsilon_0(x)$ such that $\varepsilon \leq \varepsilon_0(x)$ implies $u_\varepsilon < \psi$. But then $\varepsilon \leq \varepsilon_0(x)$ implies $\frac{(u_\varepsilon - \psi)^+}{\varepsilon} = 0$. Then $1_A \frac{(u_\varepsilon - \psi)^+}{\varepsilon} \xrightarrow[\text{a.e.}]{} 0$, thus as $1_A \frac{(u_\varepsilon - \psi)^+}{\varepsilon} \rightharpoonup 1_A \lambda$ weakly on L^2:

$$\lambda = 0 \ \text{ a.e. on } \ A = \{u < \psi\} \ . \tag{II.8}$$

Finally to prove that u satisfies I.1, we shall use a monotonicity argument. It is therefore convenient to introduce β maximal monotone operator defined by:

$$D(\beta) =]-\infty,0] \quad \begin{cases} \beta(x) = 0 & \text{if} \quad x < 0 \\ \beta(0) = [0,+\infty[\end{cases} .$$

Remark that (II.5) is equivalent to:

$$A_1 u_\varepsilon + \mu u_\varepsilon + \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ - f_1 + \beta(A_2 u_\varepsilon + \mu u_\varepsilon + \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ - f_2) \ni 0 \ , \tag{II.9}$$

or

(II.10) $\forall p \leq 0$, $p \in L^2$ $\quad \forall \tau \in \beta(p)$, $\tau \in L^2$:

$$(f_1 - A_1 u_\varepsilon - \mu u_\varepsilon - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ - \tau, \ \ A_2 u_\varepsilon + \mu u_\varepsilon + \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ - f_2 - p)_{L^2} \geq 0 \ .$$

Now passing to the limit on (II.10) we have:

$$\text{(II.11)} \quad \begin{cases} (f_1 - \tau, \ A_2 u + \mu u + \lambda - f_2 - p)_{L^2} + (-A_1 u - \mu u - \lambda, \ -f_2 - p)_{L^2} \\ \qquad \geq \underline{\lim}(A_1 u_\varepsilon + \mu u_\varepsilon, \ A_2 u_\varepsilon + \mu u_\varepsilon) + \underline{\lim} \, \|\frac{1}{\varepsilon}(u_\varepsilon - \psi)^+\|^2_{L^2} \\ \qquad\quad + \underline{\lim}\left(\sum_{i=1,2} (A_i u_\varepsilon + \mu u_\varepsilon, \ \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+)\right) . \end{cases}$$

But from Lemma II.1 we deduce that $((u,v)) = (A_1 u + \mu u, A_2 v + \mu v)$ is a continuous coercive bilinear form on H^2 and thus is weakly l.s.c. And as $(A_i u_\varepsilon + \mu u_\varepsilon, \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+)_{L^2} \geq (A_i \psi + \mu\psi, \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+)_{L^2}$, (II.11) gives:

$$\begin{cases} (f_1 - \tau, A_2 u + \mu u + \lambda - f_2 - p)_{L^2} + (-A_1 u - \mu u - \lambda, -f_2 - p)_{L^2} \\ \qquad \geq (A_1 u + \mu u, A_2 u + \mu u)_{L^2} + \|\lambda\|^2_{L^2} + \sum_{i=1,2} (A_i \psi + \mu\psi, \lambda) \end{cases}$$

or

$$(f_1 - A_1 u - \mu u - \lambda - \tau, A_2 u + \mu u + \lambda - f_2 - p)_{L^2} \geq \sum_{i=1,2} (A_i(\psi - u) + \mu(\psi - u), \lambda)_{L^2} . \tag{II.12}$$

Now consider this scalar product on the right side:

on $A = \{u < \psi\}$ $\quad \lambda = 0$ (II.8)

on $\{u = \psi\}$ if this set has a positive measure as u and ψ are in H^2:

$$A_i \psi = A_i u .$$

Thus we have:

$$\max_{i=\{1,2\}} (A_i u + \mu u + \lambda - f_i) = 0 . \tag{II.13}$$

But the combination of (II.6), (II.7), (II.8), (II.13) is:

$$\max(A_1 u + \mu u - f_1, A_2 u + \mu u - f_2, u - \psi) = 0 . \tag{I.1}$$

Remark that taking $p = A_2 u + \mu u + \lambda - f_2$ gives:

$$u_\varepsilon \xrightarrow[H^2]{} u; \quad \frac{(u_\varepsilon - \psi)^+}{\varepsilon} \xrightarrow[L^2]{} \lambda . \tag{II.14}$$

Second step: Uniqueness: Let v be another solution of (I.1):

*$u \geq v$: Indeed as $\max_{i\in\{1,2\}} (A_i v + \mu v + \frac{1}{\varepsilon}(v - \psi)^+ - f_i) \leq 0$, we have from Prop. II.2: $v \leq u_\varepsilon$.

*Uniqueness: Consider the set $B = \{v < \psi\}$

on B $\max_{i=\{1,2\}} (A_i v + \mu v - f_i) = 0 \geq \max_{i=\{1,2\}} (A_i u + \mu u - f_i)$,

on $B^C = \{v = \psi\}$ as $\psi \geq u \geq v$, $u = \psi$. Thus if B^C has a positive measure, as

$u,v \in H^2$: $\max_{i=\{1,2\}} (A_i v + \mu v - f_i) = \max_{i=\{1,2\}} (A_i u + \mu u - f_i)$.

Thus in any case: $\max\limits_{i=\{1,2\}} (A_i v + \mu v - f_i) \geq \max\limits_{i=\{1,2\}} (A_i u + \mu u - f_i)$. But that implies (Th. II.2): $v \geq u$; that is $u = v$. ■

II.5. Some properties of the solution of I.1:

In this section all the constraints ψ are supposed to be nonnegative on $\partial\Omega$.

Prop. II.3: Let $f_1, f_2 \in L^2$ (resp. g_1, g_2), $\psi \in H^2$ (resp. φ) and let u (resp. v) be the solution of problem I.1 for f_1, f_2, ψ (resp. g_1, g_2, φ). Suppose $f_1 \geq g_1$, $f_2 \geq g_2$, $\psi \geq \varphi$, then:

$$u \geq v .$$

Proof: Obvious in view of Prop. II.2. ■

We shall give some regularity results:

Theorem II.4:

1) If we suppose: $f_1, f_2 \in L^n$, $\psi \in C^0(\bar{\Omega}) \cap H^2$ then the solution u of (I.1) is in $C^0(\bar{\Omega})$.

2) If $f_1, f_2 \in L^n$ (resp. g_1, g_2), $\psi \in L^\infty \cap H^2$ (resp. φ) and u (resp. v) is the corresponding solution of (I.1); there exists a constant c depending only on Ω and on the coefficients of A_1 and A_2 such that:

$$\|u - v\|_\infty \leq (c \sum_{i=1,2} \|f_i - g_i\|_{L^n} + \|\psi - \varphi\|_{L^\infty}) . \tag{II.15}$$

3) If $f_1, f_2 \in L^\infty$ (resp. g_1, g_2) then:

$$\|u - v\|_\infty \leq \max(\frac{1}{\mu} \|f_1 - g_1\|_\infty, \frac{1}{\mu} \|f_2 - g_2\|_\infty, \|\psi - \varphi\|_\infty) . \tag{II.16}$$

Proof: 3) is immediate on the penalized problem with Theorem I.2.

*First, suppose $f_1, f_2 \in L^n$, $\psi \in W^{2,n}$:

on the set $\{u < \psi\}$ $\max\limits_{i\in\{1,2\}} (A_i u + \mu u - f_i) = 0$,

on the set $\{u = \psi\}$ (if its measure is positive): $\max\limits_{i\in\{1,2\}} (A_i u + \mu u - (A_i\psi + \mu\psi)) = 0$.

Thus $\exists \tilde{f}_1, \tilde{f}_2 \in L^n$: $\max\limits_{i\in\{1,2\}} (A_i u + \mu u - \tilde{f}_i) = 0$. That proves continuity of u (Theorem I.2).

*Now we have only to prove 2); and 1) and 2) will be proved: let us introduce w solution of (I.1) corresponding to f_1, f_2 and φ. Then we have (from Theorem I.2):

(II.17) $$\|u_\varepsilon - w_\varepsilon\|_\infty \leq \|\psi - \varphi\|_\infty .$$

Now let us consider $w_\varepsilon - v_\varepsilon$: we can always suppose (introducing $\tilde{f}_i = \max(f_i, g_i)$ and $\tilde{g}_i = \min(f_i, g_i)$) $f_i \geq g_i$ and thus $w_\varepsilon \geq v_\varepsilon$. Thus $w_\varepsilon - v_\varepsilon \geq 0$ and:

$\max\limits_{i\in\{1,2\}} (A_i w_\varepsilon + \mu w_\varepsilon - f_i) - \max\limits_{i\in\{1,2\}} (A_i v_\varepsilon + \mu v_\varepsilon - g_i) \leq 0$. Now with the same proof than in Evans-Lions [3], we deduce:

(II.18) $$\|w_\varepsilon - v_\varepsilon\|_\infty \leq c \sum_{i=1,2} \|f_i - g_i\|_{L^n} .$$

And (II.17), (II.18) give (II.15). ■

A stronger regularity result is given by:

Theorem II.5: If we suppose $f_i, f_2 \in W^{1,p}$ and $\psi \in W^{3,p}$ $(p > n)$; then the corresponding solution u of (I.1) is in $W^{1,\infty}$.

Proof: With a translation, we take $\psi = 0$; then the same proof than in Evans-Lions [3] gives the result. ■

II.6. Some related problems:

1) First we consider a "quasi-variational inequality", that is (for example):

(II.19) $$\begin{cases} \max(A_1 u + \mu u - f_1,\ A_2 u + \mu u - f_2,\ u - M(u)) = 0 , \\ u \in H^2 \cap H_0^1 \cap L^\infty, \quad u \geq 0 , \end{cases}$$

where M is defined on $H^2 \cap H_0^1 \cap L_+^\infty$ and takes the values on $L_+^\infty \cap H^2$. We suppose:

(II.20) $$\begin{cases} M(v) \geq 0 \quad \text{if} \quad v \geq 0 \\ \text{if} \quad u \leq v, \quad M(u) \leq M(v) , \end{cases}$$

(II.21) $$f_1, f_2 \in L_+^\infty ,$$

(II.22) $$\|M(u)\|_{H^2} \leq c_1(\|u\|_{H^2} + \|u\|_\infty) + c_2 .$$

(II.23) If $u^n \rightharpoonup u$ in H^2 weakly, u^n bounded in L_+^∞, and $u^n \downarrow u$ then $M(u^n) \downarrow M(u)$.

Theorem II.6: Under assumptions (II.20), (II.21), (II.22), (II.23), and if μ is large enough, there exists a maximum solution of (II.19).

Proof: Let us give the outline of the proof: we introduce (u_ε^n):

u_ε^0 is defined by: $\max\limits_{i\in\{1,2\}} (A_i u_\varepsilon^0 + \mu u_\varepsilon^0 - f_i) = 0$, $u_\varepsilon^0 \in H^2 \cap H_0^1 \cap L_+^\infty$,

u_ε^n is defined by: $\max(A_1 u_\varepsilon^n + \mu u_\varepsilon^n - f_1, A_2 u_\varepsilon^n + \mu u_\varepsilon^n - f_2) + \frac{1}{\varepsilon}[u_\varepsilon^n - M(u_\varepsilon^{n-1})]^+ = 0$, $u_\varepsilon^n \in H^2 \cap H_0^1 \cap L_+^\infty$.

*Then: $u_\varepsilon^n \underset{n}{\downarrow} u_\varepsilon$, and $u_\varepsilon^n \downarrow$ when $\varepsilon \downarrow$

(II.24) $$\|u_\varepsilon - u_\varepsilon^n\|_\infty \le \frac{1}{(1+\varepsilon\mu)^n} \|u^0\|_\infty .$$

*With *Lemma II.1 and assumption (II.22): we prove*

(II.25) $$\|u_\varepsilon^n\|_{H^2} \le c \quad [\text{independent of } n \text{ and } \varepsilon] .$$

*Then u_ε is a solution of:

(II.26) $$\max(A_1 u_\varepsilon - f_1, A_2 u_\varepsilon - f_2) + \frac{1}{\varepsilon}[u_\varepsilon - M(u_\varepsilon)]^+ = 0 .$$

Furthermore: $u_\varepsilon \downarrow u$, $u_\varepsilon \rightharpoonup u$ weakly in H^2. Then u is a solution of (II.19).

*If v is another solution of (II.19): $v \le u_\varepsilon^n \Rightarrow v \le u_\varepsilon \Rightarrow v \le u$. ■

Theorem II.7 [see Laestch [10]]: Under the assumptions of Theorem II.6, if we add:

(II.27) $$\forall\varphi \in H^2 \cap H_0^1 \cap L_+^\infty,\ \forall\alpha \in [0,1[\ \exists\ \beta \in]\alpha,1[\ : M(\alpha\varphi) \ge \beta M(\varphi) .$$

Then there exists a unique solution of (II.19).

Proof: see Laestch [10]. ■

Examples:

1) $M(u) = k + \inf\limits_F u$ where $k \in H^2 \cap L_+^\infty$, F is a finite set included in Ω. Then M satisfies (II.20), (II.22), (II.23) and if $k \ge \alpha > 0$ then M satisfies (II.27). It is worth noting that (II.19) with such a function M is a problem of impulse control of stochastic integrals.

2) $M(u) = k + \|u\|_p$ where $\xi \in H^2 \cap L^\infty$ and $p > 1$. Then M satisfies (II.20), (II.22) and (II.23). If $\xi \ge \alpha > 0$ then M satisfies (II.27). ■

3) We shall consider now a problem similar to differential games:

$$
\text{(II.28)}\quad
\begin{cases}
\psi_1 \le u \le \psi_2, \quad u \in H^2 \cap H_0^1 \\
\text{if} \quad \psi_1 < u < \psi_2 \quad \text{then} \quad \max\limits_{i\in\{1,2\}} (A_i u + \mu u - f_i) = 0 , \\
\text{if} \quad u = \psi_2 \quad \text{then} \quad \max(A_i u + \mu u - f_i) \le 0 , \\
\text{if} \quad u = \psi_1 \quad \text{then} \quad \max(A_i u + \mu u - f_i) \ge 0 .
\end{cases}
$$

(II.28) is equivalent to

$$
\text{(II.28')}\qquad \min(\max(A_1 u + \mu u - f_1,\ A_2 u + \mu u - f_2,\ u - \psi_2),\ u - \psi_1) = 0 ,
$$

or

$$
\text{(II.28")}\qquad \max(\min(A_1 u + \mu u - f_1,\ A_2 u + \mu u - f_2,\ u - \psi_1),\ u - \psi_2) = 0 .
$$

This problem is also a problem of control of stochastic integrals with optimal stopping [at least formally]. The same method than for Theorem II.1 gives:

<u>Result</u>: If $\psi_1, \psi_2 \in H^2 (\psi_1 \le 0 \le \psi_2$ on $\partial\Omega$, $\psi_1 \le \psi_2$ on $\Omega)$, if $f_1, f_2 \in L^2$ and if μ is sufficiently large, there exists <u>a</u> solution of (II.28). ∎

<u>Open problem</u>: Uniqueness of that solution.

<u>Partial result</u>: If A_1, A_2 have constant coefficients, $\psi_1, \psi_2 \in W^{2,\infty}(\Omega)$, $f_1, f_2 \in W^{2,\infty}$ and μ is sufficiently large, we have existence and uniqueness of solutions in the class $H^2 \cap H_0^1 \cap C^0(\bar{\Omega}) \cap W^{2,p}_{loc}(\Omega)$ [for some $p > n$]. In that case the solution belongs to $H^2 \cap W_0^{1,\infty}(\Omega) \cap W^{2,\infty}_{loc}$. ∎

<u>Remark</u>: We shall not prove this result here: let us indicate that the uniqueness part is not difficult [and in fact we do not need constant coefficients and such regularity on ψ_i and f_i] but to prove the existence of a solution in $W^{2,\infty}_{loc}$ is the really difficult part. ∎

III. Bellman equation for two operators: some other problems:

III.1. Reducing μ in (I.?):

Theorem III.1: We suppose that c_1, c_2 (the 0-order terms in A_1, A_2) are nonnegative, then if $f_1, f_2 \in L^2$, the two following assumptions are equivalent:

(III.1) there exists a maximum solution in $H^2 \cap H_0^1$ of: $\max\limits_{i\in\{1,2\}} (A_i u - f_i) = 0$,

(III.2) there exists a subsolution v in $H^2 \cap H_0^1$: $\max\limits_{i\in\{1,2\}} (A_i v - f_i) \le 0$. ■

Let us remark that this result gives immediately:

Corollary III.1: If $c_1, c_2 \ge 0$, $f_1, f_2 \in L^2$ then if $f_1, f_2 \ge 0$ there exist a maximum solution in $H^2 \cap H_0^1$ of:

(III.3) $$\max_{i\in\{1,2\}} (A_i u - f_i) = 0 \ . \quad ■$$

Indeed, $v = 0$ is a subsolution for nonnegative f_i.

Open problem: Assuming (III.2) and $f_1, f_2 \in L^2$ we do not know if there is uniqueness in III.3. To prove Theorem III.1, we shall use the following result, which is only an easy adaptation of a classical result [see for example Tartar [14]] in view of Theorem II.2.

Theorem III.2: Let $f_1, f_2 \in L^2$ and let μ be large enough. We denote by $\mathcal{A}u = \max(A_1 u + \mu u - f_1,\ A_2 u + \mu u - f_2)$, and by $<<$ the following order relation:

$$\begin{cases} u << v \ \text{ if } \ \mathcal{A}u \le \mathcal{A}v \ \ (\text{in } L^2) \ . \\ u, v \in H^2 \cap H_0^1 \end{cases}$$

Let $F[u]$ be a function of $x, u(x), \nabla u(x), (u_{x_i x_j}(x))$ taking its values in $\mathbb{R}$.

We suppose:

(III.4) $$\forall u \in H^2 \cap H_0^1,\ F[u] \in L^2 \ ,$$

(III.5) $$\forall u, v \in H^2 \cap H_0^1 : u << v \Rightarrow F[u] \le F[v] \ ,$$

(III.6) $$\exists u_1, u_2 \in H^2 \cap H_0^1 : u_1 << u_2,\ u_1 << Su_1,\ Su_2 << u_2 \ ,$$

where S is the application of L^2 in $H^2 \cap H_0^1$ defined by: $\mathcal{A}S\varphi = F(\varphi)$. ■

Then in $\{u_1 << u << u_2\}$ there exist a maximum and a minimum solution of:

(III.7) $$\max(A_1 u - f_1,\ A_2 u - f_2) = F[u] \ . \quad ■$$

Remark III.1: In Section III.3, we shall study some applications of Theorem III.2.

Proof of Theorem III.1: We suppose (III.2). Let μ be large enough and take $F[u] = \mu u$: (III.4) is obviously satisfied; (III.5) is satisfied because of Theorem I.2. Let us denote by $\bar{u}$ the solution of: $A_1\bar{u} = f_1$, $\bar{u} \in H^2 \cap H_0^1$; by $u_1 = v$, by $u_2 = S\bar{u}$. Then

$$ASu_1 = \mu u_1 \geq Au_1 = Av \quad \text{(because of (III.2))} ,$$

$$ASu_2 = \mu u_2 \leq \mu\bar{u} \leq A\bar{u} \quad \text{(because of the choice of } \bar{u}) ,$$

$$Au_1 \leq \mu u_1 \leq \mu\bar{u} = Au_2 ,$$

and thus (III.6) is satisfied. Thus we have u_{min}, u_{max} minimum and maximum solutions in $\{u_1 << u << u_2\}$. Now take w another solution of (III.1): we just have to prove that $u_{min} \leq w \leq u_{max}$. Define

$$\tilde{u} \text{ by } A\tilde{u} = \max(Aw, Au_1), \quad \tilde{u} \in H^2 \cap H_0^1 .$$

Then

$$AS\tilde{u} = \tilde{u} \geq \max(\mu w, \mu u_1) \geq \max(Aw, Au_1) = A\tilde{u} ,$$

$$A\tilde{u} = \max(Aw, Au_1) \leq \mu \max(w, u_1) \leq \mu\bar{u} = Au_2 ;$$

thus we can apply another time Theorem III.2, and we have $\tilde{u}_{max}$ maximum solution in $\{\tilde{u} << u << u_2\}$. But by definition of $\tilde{u}$: $\{\tilde{u} << u << u_2\} \subset \{u_1 << u << u_2\}$. Thus:

$$\tilde{u}_{max} = u_{max} \quad \text{and} \quad w << u_{max} . \quad \blacksquare$$

Another result in the line of reducing assumption on μ is the following:

Theorem III.3: We suppose: $c_1, c_2 \geq 0$ and $f_1, f_2 \in L^n$, then there exists a unique solution u in the class $H^2 \cap H_0^1 \cap L^n$ of (III.3); and $u \in C^0(\bar{\Omega})$.

Remark III.2: In a paper to appear, we prove uniqueness of solution of (III.3) in $H^2 \cap H_0^1$ only, by using stochastic methods. $\blacksquare$

Proof:

First step, proof in a special case: We suppose there exists some $\alpha > 0$ such that $c_1, c_2 \geq \alpha$. Then define S by: if $u \in C^0(\bar{\Omega})$ Su is the solution in $H^2 \cap H_0^1 \cap C^0(\bar{\Omega})$ of:

$$\max_{i=1,2} (ASu + \mu Su - f_i) = \mu u .$$

A slight modification of Evans-Lions proof in [3] gives:

$$\|Su - Sv\|_\infty \leq \frac{\mu}{\mu + \alpha} \|u - v\|_\infty .$$

Thus the result is proved in the special case considered because if u is a solution in L^n, $Su = u$ belongs to $C^0(\bar{\Omega})$ by Evans-Lions result.

Proof in the general case:

Second step: There exists w regular $(C^\infty(\bar{\Omega}))$ such that $0 < \gamma \leq w \leq 1$, $Aw \geq \beta > 0$,

where A is any differential operator: $A = -a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + b_i \frac{\partial}{\partial x_i} + c_i$

with: $\|a_{ij}\|_\infty + \|b_i\|_\infty \leq M$; $a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \quad \forall \xi \in \mathbb{R}^n$; $c_i \geq 0$.

Furthermore γ and β depend only on M, ν and Ω:

Proof: Take a point x_0 on $\partial\Omega$ which has an exterior sphere, let us denote by 0 the center of this sphere and by p the radius.

Define $w(x) = \exp[-kp^2] - \exp[-k|x|^2]$.

Then: $Aw \geq [4k^2\nu p^2 - 2k|a_{ii}| + 2\|b_i\|_\infty |x_i|k] \exp[-k|x|^2]$. For k large enough $[k \geq k_0(\nu,M,\Omega)]$: $Aw \geq \beta > 0$. ∎

Third step: Let u, v be two solutions of (III.3): $u,v \in H^2 \cap H_0^1 \cap L^n$. Then by Evans-Lions result: $u,v \in C^0(\bar{\Omega})$. Next define $\varphi_1(t)$, $\varphi_2(t)$ by: φ for all t in $[0,1]$

$$\varphi_1(t) = 1 \quad \text{if} \quad t\{A_1u + \mu u - f_1\} + (1-t)\{A_1v + \mu v - f_1\}$$

$$\geq t\{A_2u + \mu u - f_2\} + (1-t)\, A_2v + \mu v - f_2\}$$

$$= 0 \quad \text{if not}$$

and $\varphi_2(t) = 1 - \varphi_1(t)$. Then

$$|A(u-v)| = \left|\left[\int_0^1 \varphi_1(t)dt\right]\{A_1(u-v) + \mu(u-v)\} + \left[\int_0^1 \varphi_2(t)dt\right]\{A_2(u-v) + \mu(u-v)\}\right| \leq \mu|u-v|$$

but $0 \leq \int_0^1 \varphi_i(t)dt \leq 1$ and $\sum_{i=1,2} \int_0^1 \varphi_i(t)dt = 1$ on Ω. Thus by the second step, we have:

$$\|u - v\|_\infty \leq \frac{\mu\|u - v\|_\infty}{\beta + \mu\gamma} .$$

That proves uniqueness, taking $\mu \to 0$.

Fourth step: Existence: Take μ such that $0 < \frac{\mu}{\beta + \mu\gamma} < 1$; by First step for every $v \in C^0(\bar{\Omega})$ there exists a unique $u = Sv$. Now remark that third step proves:

$$\|Su - Sv\|_\infty \leq \frac{\mu}{\beta + \mu\gamma} \|u - v\|_\infty .$$

That proves existence and the Theorem. ∎

Remark III.3: It is important to remark that all regularity results in Theorem I.2 hold with only the assumptions of Theorem III.3 [because the original proofs do not need μ large].

It is also easy to see that with methods of Theorem III.3, we can prove all the results in Section II without supposing μ large in the case of $f_1, f_2 \in L^n$, $\psi \in L^\infty$.

However, in what follows, we shall keep on assuming μ large in the purpose of shortening proofs. ∎

III.2. Two remarks on Brezis-Evans proofs:

Remark III.4: It is easy to see that in the proof of Lemma II.1 of Brezis-Evans [2], if $|\Omega|$ is small enough, the assumption $\mu \geq c_1$ is not needed. That means that (III.3) can be solved for $f_1, f_2 \in L^2$ under assumptions of Theorem III.3, if $|\Omega|$ is small enough. ∎

Remark III.5: Let us consider the following problem:

$$\text{(III.8)} \qquad \begin{cases} u \in H^2 \cap H_0^1 \\ \max\limits_{i=1,2} [A_i u + \mu u + H_i(x,u,\nabla u)] = 0 \end{cases}$$

where H_1, H_2 satisfy to:

$$\text{(III.9)} \qquad \left.\begin{matrix} \forall (x,p,q) \in \Omega \times \mathbb{R} \times \mathbb{R} \\ \exists \alpha, \beta > 0 \quad \varphi \in L_+^2 \end{matrix}\right\} \ |H_i(x,p,q)| \leq \varphi + \alpha|p| + \beta|q| ,$$

(III.10) $\qquad H_i$ is continuous on p and q ,

or

(III.10') $\qquad$ if $u_n \to u$ weakly in H^2 then $H_i(x,u_n,\nabla u_n) \xrightarrow[L^2]{} H_i(x,u,\nabla u)$.

Then:

Result 1: If μ is big enough $(\mu \geq \mu_0(\alpha, \beta, A_1, A_2, \Omega))$ and H_1, H_2 satisfy to (III.9), (III.10) (or (III.10')), then (III.8) has a solution.

Outline of the proof: 1) Solve (III.8) in a finite dim. space generated by a special basis for A_2, 2) get estimates (H^2 bounds) with the help of Lemma II.1, 3) pass to the limit using a monotonicity argument. ■

Result 2: If in addition, we suppose:

(III.11) $$|H_i(x,p,q) - H_i(x,p',q')| \leq c\{|p - p'| + |q - q'|\}$$

then for μ big enough, (III.8) has a unique solution.

Outline of the proof: Use Result 1 and prove uniqueness with Lemma II.1. Or else show that:

(III.12) $$\forall w \in L^2 \quad \exists! u \in H^2 \cap H_0^1 \qquad A_2 u + H_2(x,u,\nabla u) = w .$$

Set $Kw = A_1 u + H_1(x,u,\nabla u)$; then by Lemma II.1 K is a maximal monotone coercive and Lipschitzian operator in L^2. That implies the result as in Brezis-Evans proof. ■

Examples of application:

$$H_i(x,u,Du) = \max_{\alpha \in \Lambda} \{\sum_j b_j^{i,\alpha} \frac{\partial u}{\partial x_j} + c^{i,\alpha} u - f^{i,\alpha}\}$$

where Λ is a set of parameters.

If we suppose $b_j^{i,\alpha}$, $c^{i,\alpha}$ belong to a bounded set of L^∞ and $f^{i,\alpha}$ belong to a bounded set of L^2; then (III.9), (III.10) and (III.11) are satisfied. Thus by Result 2, we have solved Bellman's equation for an infinite set of operators (but with only two possibilities on the highest order terms). Remark that we can take $\min_{\alpha \in \Lambda}$ instead of $\max_{\alpha \in \Lambda}$. This example is meaningful from the stochastic point of view, for it increases considerably the possibilities of control. ■

III.3. Some applications of Theorem III.2:

Corollary III.2: We denote by $u = \max_{i=1,2} [A_i u + \mu u - f_i]$ where $f_1, f_2 \in L^2$. Let F be a real function, nondecreasing. We suppose:

(III.13) $$|F(x)| \leq a|x| + b \quad \forall x \in \mathbb{R} ,$$

(III.14) $$\exists u_-, u_+ \in H^2 \cap H_0^1 \; \mathcal{A}u_- \leq F(u_-), \; \mathcal{A}u_+ \geq F(u_+), \; u_- \leq u_+ .$$

Then if μ is large enough, there exist a minimum and a maximum solution in $\{v \in H^2 \cap H_0^1, u_- \leq v \leq u_+\}$ of: $Au = F(u)$.

Proof: (III.4) is satisfied because of (III.13); (III.5) because F is nondecreasing. Then $u_- \leq u_+ \Rightarrow F(u_-) \leq F(u_+) \Rightarrow u_- \ll u_+$; thus (III.6) is obviously satisfied. Theorem III.2 gives a maximum solution and a minimum solution in $\{u_- \ll v \ll u_+\}$. But if v is a solution and $u_- \leq v \leq u_+$, then

$$F(u_-) \leq F(v) \leq F(u_+) \text{ and } u_- \ll v \ll u_+ . \quad \blacksquare$$

Examples:

1) Take $F(x) = a\sqrt{x} + b$ with $a > 0, b \geq 0$. Remark that if $f_1, f_2 \in L_+^\infty$: $A0 \leq F(0)$, thus we can take $u_- = 0$. Let us denote by u_λ the solution of: $Au_\lambda = \lambda$. Then if $f_1, f_2 \in L^\infty$: $\|u_\lambda\|_\infty \leq \dfrac{\lambda + \|f_1\|_\infty \quad \|f_2\|_\infty}{\mu}$, thus for λ sufficiently large $a\sqrt{u_\lambda} + b \leq \lambda$. It is now easy to show that: if $f_1, f_2 \in L_+^\infty$ there exist a minimum and a maximum solution in $\{v \in H^2 \cap H_0^1 \cap L_+^\infty\}$ which are continuous. Furthermore if $b > 0$, we have uniqueness by *Laetsch argument (see [10])*.

2) We can treat by the same techniques: existence and uniqueness of solution of

$$\begin{cases} Au = M(u) & \text{where } M(u) = k + \inf\limits_{\xi \in I(x)} u(\xi)\,(I(x) \subset \bar{\Omega}) \text{ and} \\ u \in H^2 \cap H_0^1 \cap L_+^\infty & L^\infty \ni k \geq k_0 > 0 . \end{cases}$$

Corollary III.3 [see Tartar [14]]: We denote by $Au = \max\limits_{i=1,2} \{A_i u - f_i\}$ where $f_1, f_2 \in L^2$. Let F be a Lipschitz real function. We suppose:

(III.14') $\quad \exists u_-, u_+ \in H^2 \cap H_0^1 \quad Au_- + F(u_-) \leq 0 \leq Au_+ + F(u_+); \quad u_- \leq u_+$.

Then there exist a maximum and a minimum solution in $\{v \in H^2 \cap H_0^1\ u_- \leq v \leq u_+\}$ of $Au + F(u) = 0$.

Proof: Take μ large enough: $x \overset{G}{\mapsto} \mu x - F(x)$ is *nondecreasing;* (III.13) is satisfied for G. And

$$Au_- + \mu u_- \leq \mu u_- - F(u_-) = G(u_-)$$

$$Au_+ + \mu u_+ \geq \mu u_+ - F(u_+) = G(u_+) .$$

Thus Corollary III.2 gives the result. $\blacksquare$

Example: Take $F(u) = \cos u$, we may take $Au_\pm = \pm 1$. Then it is easy to show the existence of a maximum and of a minimum solution $u \cap H^2 \cap H_0^1$ of: $Au + F(u) = 0$. $\blacksquare$

III.4. Bellman equation and Schwarz method:

The result stated in this section is proved in Lions [11], and its stochastic interpretation will be carried out in a paper to appear.

We need some notations: Let Ω_1, Ω_2 be two regular subdomains of Ω such that $\Omega_1 \cup \Omega_2 = \Omega$, $\Omega_1 \cap \Omega_2$ is nonempty. Let us denote by $\gamma_1 = \overline{\partial\Omega_1 \cap \Omega}$, $\gamma_2 = \overline{\partial\Omega_2 \cap \Omega}$; we suppose $\partial\Omega_i - \gamma_i$ nonempty. We suppose

$$\exists x_i \in \gamma_i : \sup_{x \in \gamma_i} |x - x_i| < \inf_{x \in \gamma_j} |x - x_i| \quad [(i,j) = (1,2), (2,1)] . \tag{III.15}$$

Let $f_1, f_2 \in L^1(\Omega)$, let u be the solution of [we suppose only $c_1, c_2 \geq 0$]

$$\max_{i=1,2} (A_i u - f_i) = 0; \quad u \in H^2 \cap H_0^1 . \tag{III.16}$$

Definition III.1: The sequence (u_n) of the Schwarz method [see Lions [11]] is defined by: $u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap C^0(\bar{\Omega})$; then

$$\begin{cases} \max\limits_{i=1,2} (A_i u^{2n} - f_i) = 0 \text{ in } \Omega_2;\ u^{2n} \in H^2(\Omega_2) \\ u^{2n} - u^{2n-1} \in H_0^1(\Omega_2) \quad [\text{and we extend } u^{2n} \text{ by } u^{2n-1} \text{ to } \Omega] , \end{cases} \tag{III.17}$$

[and a similar problem on Ω_1 for u^{2n+1}].

Remark III.6: To see that problem (III.17) is well posed, remark that if χ_i $(i = 1,2)$ are smooth functions on $\bar{\Omega}$ such that: $0 \leq \chi_i \leq 1$ on $\bar{\Omega}$, $\chi_1 + \chi_2 \equiv 1$ on $\bar{\Omega}$, $\chi_i \equiv 0$ on a neighborhood of $\overline{\Omega - \Omega_i}$; then $u^{2n} - u^{2n-1} \in H_0^1(\Omega_2) \iff u^{2n} - \chi_1 u^{2n-1} \in H_0^1(\Omega_2)$; and $\chi_1 u^{2n-1} \in H^2(\Omega)$. ∎

Theorem III.4: Under assumption (III.15), there exists a real k in $]0,1[$ depending only on Ω, and on the coefficients of A_1, A_2, such that:

$$\|u - u^n\|_{C^0(\bar{\Omega})} \leq k^{n-1} \|u - u^0\|_{C^0(\bar{\Omega})} . \tag{III.18}$$ ∎

Remark III.7: A similar result holds with Bellman equation for an infinite set of operators in a bounded domain of $\mathbb{R}^2$ [see Lions [11]]. ∎

III.5. Bellman equation and control of jump points:

As in the preceding section, the result stated in this section is proved in Lions [12], and its stochastic interpretation [control of jump points of stochastic integrals] will

be carried out in a paper to appear. We shall take a rather simple example: let K be a compact set included on Ω. We suppose:

(III.19) $$\exists x_0 \in K: \sup_{x \in K} |x - x_0| < \inf_{x \in \partial\Omega} |x - x_0| ,$$

(III.20) $$f_1, f_2 \in L^n(\Omega) ,$$

(III.21) $$c_1, c_2 \geq \alpha > 0 ,$$

(III.22) $$\psi \in H^2(\Omega) \cap C^0(\bar{\Omega}) .$$

Then:

Theorem III.5: Under assumptions (III.19), (III.20), (III.21) and (III.22), the condition

(III.23) $$\begin{cases} \text{if } \underline{u} \text{ is the solution of } \begin{cases} \max(A_1\underline{u} - f_1, A_2\underline{u} - f_2, \underline{u} - \psi) = 0 \text{ on } \Omega \\ \underline{u}|_{\partial\Omega} = \sup_{\partial\Omega} \psi, \quad \underline{u} \in H^2 \end{cases} \\ \text{then } \sup_K \underline{u} \leq \sup_{\partial\Omega} \psi \end{cases}$$

is a necessary and sufficient condition for the existence and uniqueness of u solving:

(III.24) $$\begin{cases} \max(A_1 u - f_1, A_2 u - f_2, u - \psi) = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = \sup_K u \\ u \in H^2 . \end{cases}$$

Then $u \in C^0(\bar{\Omega})$. ∎

Remark III.8: Remark that condition: $\{\sup_K \psi \leq \sup_{\partial\Omega} \psi\}$ implies III.23. ∎

Remark III.9: A similar result holds with Bellman equation for an infinite set of operators in a bounded domain of $\mathbb{R}^2$ [see Lions [12]]. ∎

III.6. A result of singular perturbations in Bellman equation:

We begin this section by a remark. If we suppose:

(III.25) The bilinear form $a(u,v) = (A_1 u, v)_{L^2}$ on H_0^1 is coercive.

Then a slight modification of Lemma II.1 gives:

(III.26) If $\varepsilon \leq \varepsilon_0 \ \exists\alpha > 0 (A_1 u, A_2 u + \frac{1}{\varepsilon} u)_{L^2} \geq \alpha \|u\|^2_{H^2}$.

Thus Brezis-Evans proof of Theorem I.1 gives (in particular):

(III.27) $\forall\varepsilon \leq \varepsilon_0 \ \exists! u_\varepsilon \in H^2 \cap H_0^1 \ \max(A_1 u_\varepsilon, \varepsilon A_2 u_\varepsilon + u_\varepsilon - \psi) = 0$; where

(III.28) $$\psi \in H_0^1 .$$

We denote by u the solution of the "usual" variational inequality:

$$\text{(III.29)} \qquad \begin{cases} a(u, v - u) \geq 0 \quad \forall v \leq \psi \quad v \in H_0^1 \\ u \in H_0^1, \quad u \leq \psi . \end{cases}$$

Then:

<u>Theorem III.6</u>: Under assumptions (III.25), (III.28), we have:

1) When $\varepsilon \to 0$: $u_\varepsilon \xrightarrow[H_0^1]{} u$.

2) If $\psi \in H^2$, then:

$$\text{(III.30)} \qquad \|u_\varepsilon - u\|_{H_0^1} \leq c\varepsilon^{1/3} .$$

<u>Proof</u>: 1) We have:

$$\text{(III.31)} \qquad \varepsilon(A_1 u_\varepsilon, A_2 u_\varepsilon) + a(u_\varepsilon, u_\varepsilon - \psi) = 0$$

or

$$\varepsilon(A_1 u_\varepsilon, A_2 u_\varepsilon + \varepsilon_0^{-1} u_\varepsilon) + a(u_\varepsilon, (1 - \varepsilon\varepsilon_0^{-1})u_\varepsilon - \psi) = 0 .$$

From (III.26) we deduce that u_ε is bounded in H_0^1 and $\varepsilon\|u_\varepsilon\|_{H^2}^2$ is bounded. If $u_\varepsilon \to \tilde{u}$ weakly in H_0^1 then $u_\varepsilon \leq \psi - \varepsilon A_2 u_\varepsilon \Rightarrow \tilde{u}_\varepsilon \leq \psi$. But if $v \in H_0^1$, $v \leq \psi$:

$$(A_1 u_\varepsilon, v - (1 - \varepsilon\varepsilon_0^{-1})u_\varepsilon)_{L^2} \geq \varepsilon(A_1 u_\varepsilon, A_2 u_\varepsilon + \varepsilon_0^{-1} u_\varepsilon) \geq 0 .$$

Passing to the limit, we get in a classical way: $a_1(\tilde{u}, v) \geq a_1(\tilde{u}, \tilde{u})$; thus $\tilde{u} = u$, and taking $v = u$, we obtain:

$$u_\varepsilon \xrightarrow[H_0^1]{} u .$$

2) Now suppose $\psi \in H^2$, then $u \in H^2$. First remark:

$$\text{(III.32)} \qquad a(u_\varepsilon, u - (1 - \varepsilon\varepsilon_0^{-1})u_\varepsilon) \geq 0 .$$

But

$$a(u, u_\varepsilon - u) = (A_1 u, u_\varepsilon + \varepsilon(A_2 u_\varepsilon) - u)_{L^2} - \varepsilon(A_1 u, A_2 u_\varepsilon)_{L^2}$$

then

$$a(u, u_\varepsilon - u) \geq -\varepsilon(A_1 u, A_2 u_\varepsilon)_{L^2} .$$

Combining this inequality with (III.32), we get:

(III.33) $$a(u_\varepsilon - u,\ u_\varepsilon - u) \le \varepsilon(A_1 u,\ A_2 u_\varepsilon) + c\varepsilon .$$

But we have:

$$\varepsilon(A_1 u_\varepsilon,\ A_2 u_\varepsilon + \varepsilon_0^{-1} u_\varepsilon) + a(u_\varepsilon,\ (1 - \varepsilon\varepsilon_0^{-1}) u_\varepsilon - \psi) = 0$$

and

$$a(u,\ u - \psi) = 0 ;$$

thus

$$\varepsilon \|u_\varepsilon\|^2_{H^2} \le K\{\|u_\varepsilon - u\|_{H_0^1} + \varepsilon\} .$$

With (III.33) this implies: $\|u_\varepsilon - u\|^2_{H_0^1} \le K_1 \varepsilon + K_2 \|u_\varepsilon - u\|^{1/2}_{H_0^1} \sqrt{\varepsilon}$; which gives (III.31). ∎

<u>Remark III.10</u>: The stochastic interpretation of this result is interesting: indeed a control of stochastic integrals degenerates in an optimal stopping. This point of view will be developed in a paper to appear. ∎

IV. Bellman equation for one parabolic and one elliptic operators:

IV.1. Main result:

Let T be a positive real, we shall consider an evolution problem on $[0,T] \times \Omega$: we suppose now that the coefficients of A_1 and A_2 depend on time:

(IV.1) $$a^k_{kp}, b^i_k, c^i \in C^2([0,T] \times \bar{\Omega}) \quad i \in \{1,2\}, \quad 1 \le k, p \le n ;$$

[in fact we do not need such smoothness]. We consider the following problem: (u' denotes $\frac{\partial u}{\partial t}$)

(IV.2) $$\begin{cases} \max(u' + A_1 u - f_1, A_2 u + \mu u - f_2) = 0 \quad \text{in} \quad [0,T] \times \Omega \\ u \in L^2(0,T; H^2 \cap H^1_0); \quad u' \in L^2(0,T; L^2), \quad u(0) = u_0 . \end{cases}$$

We shall need the following assumption:

(IV.3) $$\forall t \quad A_2(t) = A^*_2(t) \quad \text{that is} \quad b^2_k = \sum_p \frac{\partial a^2_{kp}}{\partial x_p} .$$

Then:

Theorem IV.1: Under assumptions (IV.1), (IV.3); if we suppose moreover

(IV.4) $$f_1, f_2 \in H^2(0,T; L^2)$$

(IV.5) $$u_0 \in H^1_0 \cap H^2; \quad A_2(0)u_0 + \mu u_0 - f_2(0) \le 0; \quad A_1(0)u_0 - f_1(0) \in H^1_0 ,$$

then if μ is sufficiently large, there exists a unique u solving (IV.2). And we have:

(IV.6) $$u' \in L^2(0,T; H^2 \cap H^1_0) \cap L^\infty(0,T; H^1_0) .$$

Remark IV.1: Let us recall what we said in the introduction: the problem (IV.2) arises in economics [see Bensoussan-Lesourne [1]]. ∎

Remark IV.2: 1) The methods of Brezis-Evans [2] and Evans-Lions [3] give for (IV.2) results of regularity. We shall not consider such results here.

2) The method of Section II [with a lot of technical difficulties] could give the following result: existence and uniqueness of u solving:

(IV.2') $$\begin{cases} \max(u' + A_1 u - f_1, A_2 u - f_2, u - \psi) = 0 \\ u(0) = u_0 . \end{cases}$$ ∎

Remark IV.3: From the stochastic point of view, (IV.2) can only be considered as a problem of control of stochastic integrals but with degenerate quasidiffusions. This seems to imply that the proper way to view (IV.2) is to consider it as a Bellman equation for two elliptic but degenerate operators on $]0,T[\times \Omega$ [see IV.3].

IV.2. Proof of Theorem IV.1:

Let us indicate how we prove Theorem IV.1:

1) Solve:

$$u'_{\varepsilon,\eta} + A_1 u_{\varepsilon,\eta} - f_1 + \beta_\eta(\varepsilon u'_{\varepsilon,\eta} + A_2 u_{\varepsilon,\eta} + \mu u_{\varepsilon,\eta} - f_2) = 0$$

(where β_η is a smooth approximation of β (see Section II.4)).

2) Estimates on $u_{\varepsilon,\eta}$, $u'_{\varepsilon,\eta}$.

3) Pass to the limit when $\varepsilon \to 0$: $u_{\varepsilon,\eta} \to u_\eta$: $u'_\eta + A_1 u_\eta - f_1 + \beta_\eta(A_2 u_\eta + \mu u_\eta - f_2) = 0$.

4) Pass to the limit when $\eta \to 0$: $u_\eta \to u$: $u' + A_1 u - f_1 + \beta(A_2 u + \mu u - f_2) = 0$.

5) Uniqueness.

As 4) is proved by the same techniques as for 3): we shall consider existence proved at the end of 3).

1) We begin by a remark: considering the change of solution: $u = ve^{+kt}$ we may suppose that if μ is large enough: $\exists \alpha > 0$ for all $v \in L^2(0,T; H^2 \cap H_0^1)$

$$\int_0^t (A_1 v, A_2 v + \mu v)_{L^2} ds \geq \alpha \int_0^t \|v\|^2_{H^2} ds \,, \tag{IV.7}$$

[it is only Lemma II.1]. Remark also that Brezis-Evans proof of Theorem I.1 in [2] give obviously: $\exists ! u_{\varepsilon,\eta}$

$$\left\{ \begin{array}{l} u'_{\varepsilon,\eta} + A_1 u_{\varepsilon,\eta} - f_1 + \beta_\eta(\varepsilon u'_{\varepsilon,\eta} + A_2 u_{\varepsilon,\eta} + \mu u_{\varepsilon,\eta} - f_2) = 0 \quad \text{in }]0,T[\times \Omega \\ u_{\varepsilon,\eta} \in L^2(0,T; H^2 \cap H_0^1);\ u'_{\varepsilon,\eta} \in L^2(0,T; L^2);\ u_{\varepsilon,\eta}(0) = u_0 \ ; \end{array} \right. \tag{IV.8}$$

where β_η is an increasing C^1 function on $\mathbb{R}$: $\beta_\eta(x) = 0$ if $x \leq 0$, $\beta_\eta(x) > 0$ if $x > 0$. We have also $u_{\varepsilon,\eta} \in L^\infty(0,T; H_0^1)$. [In fact a good choice, for our problem, of β_η is the following: $\beta_\eta(x) = \varphi(\frac{x}{\eta})$ where φ is an increasing C^1-diffeomorphism on $]0,\infty[$, φ is $C^1(\mathbb{R})$ and $\varphi(x) = 0$ if $x \leq 0$.] As it was explained above, we shall forget about η and denote u_ε the solution of (IV.8).

2) Estimates on u_ε, u'_ε:

*Estimates on u : We multiply (IV.8) by $\varepsilon u'_\varepsilon + A_2 u_\varepsilon + \mu u_\varepsilon - f_2$ and denoting by $a_1(u,v) = (\tilde{A}_1 u,v)_{L^2}$ and $a_2(u,v) = (A_2 u + \mu u,v)_{L^2}$ the bilinear forms coercive on H_0^1 [$\tilde{A}_1$ is the self-adjoint part L^2 of A_1 : $A_1 = \tilde{A}_1 + B$ where B is a pure first order operator], we have:

$$(u'_\varepsilon,\varepsilon u'_\varepsilon)_{L^2} + (u'_\varepsilon,A_2 u_\varepsilon + \mu u_\varepsilon)_{L^2} - (u'_\varepsilon,f_2)_{L^2} + (A_1 u_\varepsilon,A_2 u_\varepsilon + \mu u_\varepsilon)_{L^2}$$
$$+ \varepsilon(u'_\varepsilon,A_1 u_\varepsilon)_{L^2} - (A_1 u_\varepsilon,f_2)_{L^2} - (f_1,\varepsilon u'_\varepsilon)_{L^2} - (f_1,A_2 u_\varepsilon + \mu u_\varepsilon - f_2)_{L^2} \le 0 .$$

Integrating that relation between 0 and t; we have from (IV.7):

$$\left\{\begin{aligned}
&\varepsilon \int_0^t \|u'_\varepsilon\|^2_{L^2} ds + a_2(u_\varepsilon(t),u_\varepsilon(t)) - a_2(u_0,u_0) - \int_0^t \dot{a}_2(u_\varepsilon,u_\varepsilon) ds - [(u_\varepsilon,f_2)_{L^2}]_0^t \\
&\qquad + (u_\varepsilon,f'_2)_{L^2} - (f_1,\varepsilon u'_\varepsilon)_{L^2} + \alpha \int_0^t \|u_\varepsilon\|^2_{H^2} ds \\
&\qquad + \varepsilon a_1(u_\varepsilon(t),u_\varepsilon(t)) - \varepsilon a_1(u_0,u_0) + \int_0^t \varepsilon(u'_\varepsilon,Bu_\varepsilon)_{L^2} ds - \varepsilon \int_0^t \dot{a}_1(u_\varepsilon,u_\varepsilon) ds \\
&\qquad - (A_1 u_\varepsilon,f_2)_{L^2} - (f_1,A_2 u_\varepsilon + \mu u_\varepsilon - f_2)_{L^2} \le 0 ,
\end{aligned}\right.$$

where

$$\dot{a}_i(u,v) = \int_\Omega \sum_{k\ell} \alpha^{i'}_{k\ell} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k} dx + \int_\Omega c^{i'} uv\, dx$$

and $\alpha^i_{k\ell}, c^i$ are the coefficients of $a_i(\cdot,\cdot)$. Then using several times Schwarz inequality, we obtain:

$$\varepsilon \int_0^t \|u'_\varepsilon\|^2_{L^2} ds + \|u_\varepsilon(t)\|^2_{H_0^1} + \int_0^t \|u_\varepsilon\|^2_{H^2} ds \le c_1 + c_2 \int_0^t \|u_\varepsilon(t)\|^2_{H_0^1} ds .$$

Then from Grönwall inequality we have:

(IV.9) $$\int_0^T \|u_\varepsilon\|^2_{H^2} ds \le C$$

(IV.10) $$\forall t \|u_\varepsilon(t)\|_{H_0^1} \le C$$

(IV.11) $$\varepsilon \int_0^T \|u'_\varepsilon\|^2_{L^2} ds \le C .$$

But (IV.11) does not give enough estimation on u_ε' to pass to the limit.

*Estimates on $u_\varepsilon'(0)$: from the equation as $A_2(0)u_0 + \mu u_0 - f_2(0) \le 0$ and $\beta_n(x) = 0$ if $x \le 0$, we have:

$$u_\varepsilon'(0) = f_1(0) - A_1(0)u_0 \in H_0^1 \quad \text{(by (IV.5))} .$$

*Estimates on u_ε': We are going to differentiate in time (IV.8): we set $v_\varepsilon = u_\varepsilon'$

$$\text{(IV.12)} \quad \begin{cases} v_\varepsilon' + A_1 v_\varepsilon + \dot{A}_1 u_\varepsilon - f_1' + \beta'(\varepsilon u_\varepsilon' + A_2 u_\varepsilon + \mu u_\varepsilon - f_2)\{\varepsilon v_\varepsilon' + A_2 v_\varepsilon + \mu v_\varepsilon + \dot{A}_2 u_\varepsilon - f_2'\} = 0 \\ v_\varepsilon(0) = u_\varepsilon'(0) . \end{cases}$$

Here $\dot{A}_i$ are the operators, whose coefficients are the time derivatives of those of A_i. Now multiplying (IV.12) by $\{\varepsilon v_\varepsilon' + A_2 v_\varepsilon + \mu v_\varepsilon + \dot{A}_2 u_\varepsilon - f_2'\}$ and integrating between 0 and t, it is easy to obtain estimates (IV.13), (IV.14) if we remark that:

$$\int_0^t (v_\varepsilon', \dot{A}_2 u_\varepsilon)_{L^2} ds = (v_\varepsilon(t), \dot{A}_2 u_\varepsilon(t))_{L^2} - (u_\varepsilon'(0), \dot{A}_2 u_0)_{L^2}$$
$$- \int_0^t (v_\varepsilon, \dot{A}_2 v_\varepsilon)_{L^2} ds - \int_0^t (v_\varepsilon, \ddot{A}_2 u_\varepsilon)_{L^2} ds ,$$

where $\ddot{A}_2$ is the operator with coefficients which are the second time derivatives of those of A_2; and that:

$$|(v_\varepsilon(t), \dot{A}_2 u_\varepsilon(t))_{L^2}| \le p\|v_\varepsilon(t)\|^2_{H_0^1} + \frac{K}{p}\|u_\varepsilon(t)\|^2_{H_0^1} \quad \text{for all } p > 0 .$$

Thus, we have:

$$\text{(IV.13)} \qquad \int_0^T \|v_\varepsilon\|^2_{H^2} ds \le C$$

$$\text{(IV.14)} \qquad \forall t \|v_\varepsilon(t)\|^2_{H_0^1} \le C .$$

3) Passing to the limit: We shall use a classical monotonicity and compactness argument: we extract a subsequence still denoted by u_ε such that:

$$\begin{cases} u_\varepsilon \rightharpoonup u \quad \text{in } L^2(0,T;H^2 \cap H_0^1) \text{ weak, } L^\infty(0,T;H_0^1) \text{ weak * ,} \\ u_\varepsilon' \rightharpoonup u' \quad \text{in } L^2(0,T;H^2 \cap H_0^1) \text{ weak, } L^\infty(0,T;H_0^1) \text{ weak * .} \end{cases}$$

We just have to prove that u satisfies to:

(IV.15) $$u' + A_1 u - f_1 + \beta_\eta(A_2 u + \mu u - f_2) = 0 .$$

But let $\varphi \in L^2(0,T;L^2)$ then $\beta_\eta(\varphi) \in L^2(0,T;L^2)$ and:

(IV.16) $$\int_0^T (f_1 - A_1 u_\varepsilon - u'_\varepsilon - \beta_\eta(\varphi),\ \varepsilon u'_\varepsilon + A_2 u_\varepsilon + \mu u_\varepsilon - f_2 - \varphi)_{L^2} ds \geq 0 .$$

We pass to the limit:

$$\int_0^T (f_1 - \beta_\eta(\varphi),\ A_2 u + \mu u - f_2 - \varphi)_{L^2} + (-A_1 u - u', -f_2 - \varphi)_{L^2} ds \geq \overline{\lim} \int_0^T (u'_\varepsilon + A_1 u_\varepsilon, A_2 u_\varepsilon + \mu u_\varepsilon)_{L^2} ds .$$

But $\int_0^T (A_1 u, A_2 v + \mu v)_{L^2} ds$ is a coercive continuous bilinear form on $L^2(0,T;H^2)$ because of (IV.7): thus

$$\overline{\lim} \int_0^T (u'_\varepsilon + A_1 u_\varepsilon,\ A_2 u_\varepsilon + \mu u_\varepsilon)_{L^2} ds \geq \overline{\lim} \int_0^T (u'_\varepsilon, A_2 u_\varepsilon)_{L^2} ds + \int_0^T (A_1 u,\ A_2 u + \mu u)_{L^2} ds ,$$

and

$$\int_0^T (u'_\varepsilon,\ A_2 u_\varepsilon)_{L^2} ds = a_2(u_\varepsilon(T),\ u_\varepsilon(T)) - a_2(u_0, u_0) - \int_0^T \dot{a}_2(u_\varepsilon(s),\ u_\varepsilon(s)) ds .$$

But by compactness argument $u_\varepsilon \to u$ in $L^2(0,T;H_0^1)$ strong, and thus

$$\int_0^T \dot{a}_2(u_\varepsilon(s),\ u_\varepsilon(s)) ds \to \int_0^T \dot{a}_2(u(s),\ u(s)) ds .$$

And we have finally:

(IV.17) $$\int_0^T (f_1 - A_1 u - u' - \beta_\eta(\varphi),\ A_2 u + \mu u - f_2 - \varphi)_{L^2} ds \geq 0 .$$

That means that u is a solution of (IV.15).

4) Uniqueness: Let u_1, u_2 be two solutions. Then multiplying the corresponding equations by $A_2(u_1 - u_2) + \mu(u_1 - u_2)$ and integrating over $[0,t]$, we have:

$$\| (u_1 - u_2)(t) \|^2_{H_0^1} + \int_0^t \| u_1 - u_2 \|^2_{H^2} ds \leq C \int_0^t \| (u_1 - u_2)(s) \|^2_{H_0^1} ds .$$

Then, by Grönwall lemma, we have:

$$u_1 = u_2 \,. \quad \blacksquare$$

Remark IV.4: We see that the proof of uniqueness does not require (IV.4), (IV.5) (but only $u_0 \in H_0^1$).

IV.3. Degenerate operators in Bellman equation: some examples:

1) Bellman equation with degenerate parabolic or elliptic operators: Let us give some examples which can be treated by methods similar to that of IV.2.

Example 1:

$$\text{(IV.18)} \qquad \begin{cases} \mu u + \max\left(x \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x \partial x} - f_1, \frac{\partial u}{\partial t} - \Delta u - f_2\right) = 0 \text{ in }]0,T[\times]0,x_0[\times \mathbb{R}^{n-1} \\ u(0) = u_0 \end{cases}$$

where $x \in]0,x_0[$ and $x_0 \le +\infty$.

Example 2:

$$\text{(IV.19)} \qquad \begin{cases} \mu u + \max\left(-\Delta u - f_1, -\frac{\partial^2 u}{\partial x \partial x} - f_2\right) = 0 \text{ in }]x_0,x_1[\times \mathbb{R}^{n-1} \\ u(x_0) = u(x_1) = 0 \end{cases}$$

where $x \in]x_0,x_1[$ $-\infty \le x_0 < x_1 \le +\infty$.

Example 3:

$$\text{(IV.20)} \qquad \begin{cases} \mu u + \max\left(a(x) \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x \partial x} - f_1, -\Delta u - f_2\right) = 0 \text{ in }]x_0,x_1[\times \mathbb{R}^{n-1} \\ u(x_0) = u(x_1) = 0 \end{cases}$$

where $x \in]x_0,x_1[$ $-\infty \le x_0 < x_1 \le +\infty$; $a \in L^\infty$, $\frac{da}{dx} \in L^\infty$.

In these examples, a method is to solve approximate problems [for example

$$\text{(IV.19')} \qquad \mu u_\varepsilon + \max\left(-\Delta u_\varepsilon - f_1, -\frac{\partial^2 u_\varepsilon}{\partial x \partial x} - \varepsilon \Delta u_\varepsilon - f_2\right) = 0] \,,$$

then obtain estimates from the equation, then differentiate the equation to obtain new estimates and pass to the limit. Remark that in Example 1, we have to work with spaces with weight.

2) <u>An evolution problem of higher order</u>: We consider two differential coercive isomorphisms self-adjoint from H^m into H^{-m}: A_1, A_2. We solve: for $\lambda > 0$

$$(IV.21) \qquad \begin{cases} \max(u' + A_1 u - f_1,\ \lambda u'' + \lambda A_1 u' + A_2 u - f_2 u) = 0 & \text{in }]0,T[\times \Omega \\ u(0) = u_0,\ u'(0) = v_0 . \end{cases}$$

A method is to solve (IV.21) in a finite dim. space [in a special basis of A_1], then have estimates (direct and with differentiation of IV.21), then pass to the limit. Remark that in the case of $m = 1$, we can take $\lambda = 0$ [Theorem IV.1].

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are proved

We prove existence, uniqueness and regularity properties for a solution u of the control problem:

sub 1 sub 2 psi Omega del Omega

$$\begin{cases} \max(A_1 u - f_1, A_2 u - f_2, u - \psi) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

as well as related problems, where A_1, A_2 are two second-order uniformly elliptic operators. Our methods are based upon monotonicity arguments.

sub 2

sub 1

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